

Non-self-similar behavior in the LSW theory of Ostwald ripening

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Abstract

The classical Lifshitz-Slyozov-Wagner theory of domain coarsening predicts asymptotically self-similar behavior for the size distribution of a dilute system of particles that evolve by diffusional mass transfer with a common mean field. Here we consider the long-time behavior of measure-valued solutions for systems in which particle size is uniformly bounded, i.e., for initial measures of compact support.

We prove that the long-time behavior of the size distribution depends sensitively on the initial distribution of the largest particles in the system. Convergence to the classically predicted smooth similarity solution is impossible if the initial distribution function is comparable to any finite power of distance to the end of the support. We give a necessary criterion for convergence to other self-similar solutions, and conditional stability theorems for some such solutions. For a dense set of initial data, convergence to any self-similar solution is impossible.

Key words: Kinetics of phase transitions, domain coarsening, asymptotic behavior, self-similarity, stability, chaos.

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1 Introduction

The problem of Ostwald ripening is to explain the growth of the typical particle size observed in the late-stage aging of heterogeneously nucleated phase transitions. In the classical theory of Lifshitz and Slyozov⁽⁹⁾ and Wagner,⁽¹⁷⁾ particles of the minority phase are assumed to be widely separated spheres that interact only through quasi-static diffusional exchange with a mean field. The chemical potential is proportional to curvature on the particle boundaries, and its gradient provides the flux that determines the particle growth rate. For background to the problem and a review of the physical literature, see refs. 15, 16.

In the dilute approximation with appropriate units, the volume v of any particle obeys the evolution law

$$\frac{\partial v}{\partial t} = \Lambda(v, \theta(t)) \equiv v^{1/3} \theta(t) - 1, \quad (1)$$

where $\theta(t)$ is the same for all particles. The critical volume is $\theta(t)^{-3} = \frac{4\pi}{3} R_c(t)^3$, where $R_c(t)$ is the critical radius. Conservation of mass determines the value of $R_c(t)$ or $\theta(t)$, and if mass in the diffusion field can be neglected, total particle volume is conserved and one finds that the critical radius equals the average radius of currently existing particles. Particles with radius larger than $R_c(t)$ are growing, while small particles shrink to zero size in finite time and disappear.

The LSW theory is concerned with a long-time regime in which most initially existing particles have disappeared but a large number remain. The system is characterized by the particle size distribution $f(t, v)$, which we consider normalized so that $\int_0^v f(t, u) du$ is the number of remaining particles with volume less than v , divided by the initial number at time 0. As particles of finite size are neither created nor destroyed, $f(t, v)$ should satisfy the conservation law

$$\partial_t f + \partial_v (\Lambda(v, \theta(t)) f) = 0, \quad (2)$$

where

$$\theta(t) = \int_0^\infty f(t, v) dv \bigg/ \int_0^\infty v^{1/3} f(t, v) dv. \quad (3)$$

(These LSW equations also arise from a different model of particle clustering, the Becker-Döring equations, as recently shown by Penrose⁽¹³⁾.)

Lifshitz and Slyozov⁽⁹⁾ and Wagner⁽¹⁷⁾ argued that for large times, the size distribution should behave in a self-similar manner. They predicted:

- (a) The critical radius cubed grows linearly for large time, like $4t/9$.
- (b) The particle size distribution approaches a form which is self-similar under scaling by the critical radius; and
- (c) In general, all scaled size distributions will approach the same self-similar solution. This solution is smooth, compactly supported and explicitly computable.

The basic structure of the scaling predictions (a) and (b) have been validated in

experiments, but the form predicted in (c) was not found. (Also, the particular rate in (a) has proved difficult to measure.) Breakdown of the law (2) (which implies that all particles of a given size have exactly the same growth trajectory) is the most likely route for failure of the theory to explain experiment, as has long been recognized. See ref. 11 for a mathematical justification of this law by homogenization methods, that suggests that it is not necessarily valid, even in the limit of small volume fraction, if the electrostatic *capacity* of the particles does not also vanish.

In this paper we address the mathematical problem of whether solutions of (2) in fact must have the asymptotically self-similar form predicted by the LSW theory. In the LSW analysis it is clear that the equations permit a family of self-similar solutions that are not infinitely smooth at their point of vanishing. However, the arguments of Lifshitz and Slyozov suggest that these are unstable.

The present results indicate that the long-time behavior of the scaled particle size distribution depends sensitively on the behavior of the initial distribution near the end of its support. Let $\bar{v}(t)$ denote the supremum of the support of the volume distribution; we assume $\bar{v}_0 := \bar{v}(0) < \infty$ throughout this paper. We prove, for example, that if the initial distribution has the property that

$$\psi_0(u) := \int_{\bar{v}_0-u}^{\bar{v}_0} f(0, v) dv \geq au^p \quad (u \geq 0) \quad (4)$$

for *some* exponent $p < \infty$ and some $a > 0$, then convergence to the smooth self-similar solution predicted by the LSW theory is impossible.

The non-smooth self-similar solutions are characterized by their exponent of vanishing. Given any exponent p in $(0, \infty)$ there is a unique (up to scaling) self-similar solution satisfying $\psi_0(u) \sim au^p$ as $u \rightarrow 0$. Our main result is to establish a condition on the initial data that is *necessary* for convergence to these self-similar distributions. The condition is a little weaker than requiring that $\lim_{u \rightarrow 0} \psi_0(u)/u^p$ exists — it states that the map ψ_0 must be “regularly varying at 0 with exponent p ” (see Definition 5.5 and the books of Seneta⁽¹⁴⁾ and Bingham et al.⁽²⁾).

We conjecture that this necessary condition is also *sufficient* for convergence to a self-similar distribution characterized by a finite value of p . Numerical simulations that we have performed tend to support this conjecture. We prove it for initial data sufficiently close in a certain sense to some self-similar solution corresponding to a sufficiently small value of p . More generally, given arbitrary initial data that satisfy the necessary condition, we prove that the solution converges to self-similar form *if* the scaled inverse critical radius $\bar{v}(t)^{1/3}\theta(t)$ converges as $t \rightarrow \infty$.

By making an arbitrarily small change to any given initial distribution in the vicinity of the tip of its support, one can easily arrange that the convergence criterion holds, or does not hold, at will. Our results then imply that there is a dense set of initial data for which convergence to *any* self-similar solution is impossible. If it is true that our necessary condition is also sufficient, then there is also a dense set of data that yield convergence to any given self-similar solution corresponding to a finite value of p .

The basic idea underlying these results is a simple one. That is the notion that

the essential effect of time evolution is to *stretch* the size distribution, *chop off* the part corresponding to “dead” particles, and *normalize* to keep total volume constant. In other words, the action of the dynamics (up to a smooth distortion) is simply to “zoom in” as with a microscope on the details of the particle size distribution near the tip of its support.

If the initial data behave in an irregular fashion near this tip on a microscopic scale, we can expect the solution to exhibit irregular behavior in time on a macroscopic scale, as particles are extinguished in order of increasing size and the distribution is normalized to conserve total volume. Indeed, the analysis in the proof of Theorem 5.10 below shows that, with respect to the logarithm of distance to the tip of support, the dynamics near the tip is rather well described by a continuous shift map. This suggests that the dynamics determined by the LSW model (2) may actually be *chaotic* in a mathematical sense (with a dense trajectory and a dense set of time-periodic solutions).

It would be wrong to infer, however, that the dynamics of Ostwald ripening is chaotic in any meaningful physical sense. The mechanism that produces sensitive dependence of long-time behavior on initial data in the LSW model depends heavily on the fact that the growth rate is the same for all particles of the same size. Lifshitz and Slyozov⁽⁹⁾ already assumed some kind of breakdown in this rule in arguing that the non-smooth self-similar distributions would not be observed physically due to the effect of “encounters between different grains.” It is plausible that a theory that seeks to quantitatively explain the self-similar distributions observed in Ostwald ripening should account for dispersion of growth rates among same-size particles.

The notion that the asymptotic form of the size distribution in LSW theory depends on the precise mode of vanishing of the initial distribution has emerged recently (independently of the present work) in work of Meerson & Sasorov,⁽¹⁰⁾ Giron et al.,⁽⁸⁾ and Carr and Penrose.⁽⁶⁾ The sole mathematical work in this connection is ref. 6, which concerns a simplified LSW theory for which the volume growth rate is modeled by $v\theta(t) - 1$, yielding linear characteristic equations. Carr and Penrose prove convergence to that self-similar solution characterized by $\psi_0(u) \sim au^p$ for initial data that have the property that $\int_0^x \psi_0(u) du = A(x)x^{p+1}$, where $p > 0$ and A is differentiable with $A(0) > 0$. Furthermore, they give examples of initial data for which no convergence to self-similar form occurs, and they also have results for data of infinite support.

Giron et al.⁽⁸⁾ expand a physical discussion given by Meerson & Sasorov⁽¹⁰⁾, and treat a class of models in which $F(R, t)$, the number density of domains with radius R at time t , evolves according to a law of the form

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial R}(VF) = 0, \quad V(R, t) = \frac{1}{R^n} \left(\frac{1}{R_c(t)} - \frac{1}{R} \right) \quad (n \geq -1).$$

The critical radius $R_c(t)$ evolves so that the total volume $\int_0^\infty R^3 F(R, t) dR$ is conserved. These authors assert that a size distribution initially describable by a power law $A_0(R_m(0) - R)^\lambda$ will for any $t > 0$ have an expansion in which the leading term has the form $A(t)(R_m(t) - R)^\lambda$, where the exponent λ is invariant in time. It is then asserted that if a self-similar asymptotic regime is ever reached, it must correspond to the self-

similar solution having the same $\lambda \in (0, \infty)$. This assertion is supported by numerical computation and a linearized asymptotic analysis at the tip of support, for initial data having the form $F_0(R) = \xi^\lambda g_1(\xi) + \xi^{\lambda_2} g_2(\xi)$ for $\xi = R_m(0) - R > 0$, where g_1 and g_2 are analytic and $\lambda_2 > \lambda > 0$. (The condition $\lambda > 0$ corresponds to $p > 1$ in the present paper.)

In the metallurgical literature, Brown (see refs. 3–5 and the references therein) had earlier asserted that an arbitrary initial distribution evolves to that self-similar solution “with the same tail,” but by this he appears to always mean infinite-support self-similar solutions, for which the volume $\int_0^\infty v f(t, v) dv$ is infinite. Regarding distributions of compact support (which he calls “discontinuous”), he asserts they evolve to the distribution predicted by LSW. E.g., in ref. 4 he says “Both Hillert et al. (4) and Brown (5) have shown that an initial triangular distribution develops into the LSW distribution, albeit slowly.” This kind of assertion is contradicted by the results we prove in this paper. However, Brown does make a number of persuasive arguments regarding the physical meaning of perturbing the distribution in the range of the largest particles in the system, and the ultimate dominance of the distribution of the largest particles. For example, in ref. 5 he mentions that the *rate* of coarsening can be slowed if the distribution favors the largest particles (as it does for self-similar solutions corresponding to small values of the exponent p).

The structure of this paper is as follows. In Section 2 we reformulate the evolution law for the size distribution and recall the results proved in ref. 12 concerning the well posedness of the initial value problem. We show that the solution is given for finite time by an analytic distortion of the initial data. Section 3 contains results for long time that involve the unscaled variables. In particular, if the initial size distribution contains a Dirac delta at the tip of the support, the distribution converges to a stationary Dirac delta distribution as $t \rightarrow \infty$, and if not, then $\bar{v}(t)$ increases to infinity. In Section 4, we classify self-similar solutions by exponent of vanishing. In Section 5 we scale the distribution function by the maximum particle volume $\bar{v}(t)$ and establish a variety of results as indicated above, concerning long-time convergence in the rescaled variables. We conclude in Section 6 with a discussion, and give examples of initial data yielding convergence and nonconvergence to self-similar form.

2 Well posedness, regularity, stretching

2.1 A well-posed formulation

Some care should be taken in the physical interpretation of stability and instability results asserted for particle size distributions. One ought to consider what kinds of particle volume distributions are physically meaningful, and what kinds of perturbations make sense. It is plausible that in late-stage coarsening it is not possible to nucleate particles that are indefinitely large, for example. And it seems reasonable to imagine that small perturbations to the system would involve small changes to particle volumes.

In this article we treat solutions for which the initial size distribution $f(0, v) dv$ is an arbitrary probability measure with compact support. This admits distributions for which

a finite fraction of the particles all have the same size, for example. Compact support corresponds to the requirement that particle sizes initially are bounded.

A theory for well posedness of the initial value problem for such solutions is developed in ref. 12. (For a family of closely related Lifshitz-Slyozov models, Collet and Goudon⁽⁷⁾ have recently studied the initial-value problem for initial distributions $f(0, \cdot)$ that are integrable with finite first moment.) To discuss the results of ref. 12, it is better to work with an equivalent formulation using different variables. First, let $\varphi(t, v)$ denote the fraction of initially existing particles with volume $\geq v$ at time t . With $F(v) = 1 - \varphi(t, v)$, F is the usual distribution function for the particle volume distribution. When there is no danger of confusion we also refer to $\varphi(t, \cdot)$ as the “distribution function φ .” The function $v \mapsto \varphi(t, v)$ is decreasing and left-continuous at jumps with $\varphi(t, 0) = 1$. (To be precise, we say φ is *decreasing* if $\varphi(x_1) \leq \varphi(x_2)$ whenever $x_1 \geq x_2$, and similarly with *increasing*. A decreasing function need not be strictly decreasing.) The particle volume distribution is a measure, in general, given formally by $f(t, v) dv = dF(v)$, that is, $\varphi(t, x) = \int_x^\infty f(t, v) dv$.

Next we invert the relationship between v and φ and consider the volume v at time t as a function of $\varphi \in [0, 1]$, taken so that $\varphi \mapsto v(t, \varphi)$ is decreasing and right-continuous at jumps with $v(t, 1) = 0$. For a finite system of particles with volumes ranked in decreasing order $v_0(t) \geq \dots \geq v_{N-1}(t)$, we have $v(t, \varphi) = v_j$ for $\varphi \in [\frac{j}{N}, \frac{j+1}{N})$. In general,

$$v(t, x) = \sup\{y \mid \varphi(t, y) > x\} \quad \text{for } 0 \leq x < 1 = \max \varphi. \quad (5)$$

We shall call the map $\varphi \mapsto v(t, \varphi)$ the *volume ranking* for the system at time t , and refer to φ as the *particle rank*. The volume ranking determines the distribution function according to the prescription

$$\varphi(t, y) = \sup\{x \mid v(t, x) > y\} \vee 0 = \begin{cases} \sup\{x \mid v(t, x) > y\} & \text{for } 0 \leq y < \max v(t, x), \\ 0 & \text{for } y \geq \max v(t, x). \end{cases} \quad (6)$$

The notion of distance between size distributions employed in the well-posedness theory of ref. 12 corresponds to the supremum-norm distance between the associated volume rankings. This is the least “maximal volume change” required to alter one size distribution into the other. Mathematically this corresponds to the notion of the L^∞ Wasserstein distance between probability measures, as shown in ref. 12.

The equation governing the evolution of the volume ranking $v(t, \varphi)$ is just (1). In ref. 12, we proved the following theorem on the global well-posedness of the initial value problem. Let rcd be the set of right-continuous decreasing functions $v_0: [0, 1] \rightarrow \mathbb{R}$ with $v_0(1) = 0$. On rcd we have the metric topology given by the supremum norm of the difference:

$$\|v_1 - v_2\| = \sup_{\varphi} |v_1(\varphi) - v_2(\varphi)|.$$

For any $T > 0$, $C([0, T], rcd)$ is the metric space of continuous $v: [0, T] \rightarrow rcd$, with metric given by $\sup_{[0, T]} \|v_1(t, \cdot) - v_2(t, \cdot)\|$.

Theorem 2.1 *Given $v_0 \in rcd$, there is a unique continuous $v: [0, \infty) \rightarrow rcd$ and a unique $\theta \in L_{\text{loc}}^\infty(0, \infty)$ such that*

$$\int_0^1 v(t, \varphi) d\varphi = \int_0^1 v_0(\varphi) d\varphi$$

for all $t \geq 0$, and

$$v(t, \varphi) = v_0(\varphi) + \int_0^t (v(s, \varphi)^{1/3} \theta(s) - 1) ds$$

for all (t, φ) such that $v(t, \varphi) > 0$. Moreover, the map from v_0 to v is locally Lipschitz continuous from rcd into $C([0, T], rcd)$, for any $T > 0$.

The value of $\theta(t)$ is determined by conservation of volume to be

$$\theta(t) = \bar{\varphi}(t) \left/ \int_0^{\bar{\varphi}(t)} v(t, \varphi)^{1/3} d\varphi \right. \quad (7)$$

where $\bar{\varphi}(t)$ is the end of the support of $v(t, \cdot)$, i.e.,

$$\bar{\varphi}(t) = \sup\{\phi \in [0, 1] \mid v(t, \phi) > 0\}. \quad (8)$$

The function $\bar{\varphi}$ is decreasing in time, but may have jumps. Consequently θ can jump, but it is positive and bounded on finite intervals of time. For fixed φ , $t \mapsto v(t, \varphi)$ is Lipschitz continuous, and satisfies (1) for almost every t , as long as $v(t, \varphi) > 0$.

2.2 Stretching

The result below indicates that for finite time the action of the dynamics *stretches* the volume ranking vertically in a smooth manner (by an analytic map). Thus, we can anticipate that whatever limiting behavior or irregularity is present in the initial data $v_0(\varphi)$ in the limit $\varphi \rightarrow 0$ will be preserved for any finite length of time.

Proposition 2.2 *Let (θ, v) be a solution of (1) as given by Theorem 2.1. For $x > 0$, let $\mathcal{V}(t, x)$ be the solution of*

$$\mathcal{V}(t, x) = x + \int_0^t (\mathcal{V}(s, x)^{1/3} \theta(s) - 1) ds$$

defined on the maximal time interval $[0, \hat{t}(x))$ where $\mathcal{V} > 0$. Then

- (a) \mathcal{V} is analytic in x , i.e., whenever $\mathcal{V}(t_0, x_0) > 0$, the map $x \rightarrow \mathcal{V}(t_0, x)$ is analytic near x_0 .
- (b) $\partial \mathcal{V} / \partial x$ is strictly increasing in time for each x .
- (c) $v(t, \varphi) = \mathcal{V}(t, v_0(\varphi))$.

Proof: The formula in (c) follows by the uniqueness of solutions to the initial value problem for (1).

Since the function $w \mapsto w^{1/3}$ is smooth for $w > 0$, a standard proof shows that $x \mapsto \mathcal{V}(t_0, x)$ is continuously differentiable to any order, and each derivative is Lipschitz continuous in time. Almost everywhere in time, the derivative satisfies

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{V}}{\partial x}(t, x) = \frac{\theta(t)}{3\mathcal{V}(t, x)^{2/3}} \frac{\partial \mathcal{V}}{\partial x}(t, x), \quad \frac{\partial \mathcal{V}}{\partial x}(0, x) = 1,$$

therefore

$$\frac{\partial \mathcal{V}}{\partial x}(t, x) = \exp \int_0^t \frac{\theta(s)}{3\mathcal{V}(s, x)^{2/3}} ds,$$

and this is strictly increasing in time. This proves (b).

To prove (a) we use a result related to a theorem of Bernstein. We recall that a smooth function $g: (0, \infty) \rightarrow \mathbb{R}$ is called *completely monotone* if its derivatives satisfy $(-1)^k g^{(k)} \geq 0$ for all integers $k \geq 0$. As a consequence of Bernstein's theorem that every completely monotone function has a representation as the Laplace transform of a positive measure, it follows that every completely monotone function has an analytic extension to the open right half of the complex plane. (See ref. 1, for example, for a proof.)

We note that if $\mathcal{V}(t_0, x_0) > 0$, then it is easy to show that $\mathcal{V}(t_0, x_0) < \mathcal{V}(t_0, x) < \infty$ for all $x > x_0$. Thus, to prove (a) it suffices to show that if $\mathcal{V}(t_0, x_0) > 0$, then the function $x \mapsto (\partial \mathcal{V} / \partial x)(t_0, x_0 + x)$ is completely monotone. To do this, we show that the function given by

$$\omega(t, x) = -\mathcal{V}(t, -x + x_0) \quad \text{for } 0 \leq t \leq t_0, x < 0,$$

satisfies $\partial_x^k \omega \geq 0$ for all integers $k \geq 1$. This is true for $k = 1$ by (b). Suppose it holds for $1 \leq k < j$ for some j . Let $G(y) = -(-y)^{1/3}$ for $y < 0$, and note that the derivatives $G^{(k)}$ are positive for all $k \geq 1$. The function $\partial_x^j \omega$ satisfies

$$\frac{\partial}{\partial t} \partial_x^j \omega = \theta(t) \partial_x^j (G(\omega)) = \theta(t) (G'(\omega) \partial_x^j \omega + R_j(t, x)),$$

where $R_j(t, x)$ is a sum of terms of the form $c_\alpha G^{(k)}(\omega) \partial_x^{\alpha_1} \omega \dots \partial_x^{\alpha_n} \omega$ which have positive coefficients and involve derivatives $\partial_x^k \omega$ of order $\leq j-1$. Then it follows that $R_j(t, x) \geq 0$ and hence

$$\partial_x^j \omega(t, x) = \int_0^t \exp \left(\int_s^t G'(\omega(r, x)) dr \right) \theta(s) R_j(s, x) ds \geq 0$$

for $0 \leq t \leq t_0, x < 0$. We obtain the desired result by induction. \square

2.3 Evolution of the distribution function

For the main part of our study of long-time behavior we will work with the distribution function $\varphi(t, v)$, rescaled appropriately to study the development of self-similar form. To determine the time evolution of $\varphi(t, v)$, we need to use (6) to get the appropriate generalized inverse function. The following result describes the evolution and implies

that it is a solution of the advection equation

$$\partial_t \varphi + \Lambda(v, \theta(t)) \partial_v \varphi = 0 \quad (9)$$

whenever the initial data $\varphi_0 = \varphi(0, \cdot)$ is differentiable. The characteristics of (9) are given by the function \mathcal{V} of Proposition 2.2.

Lemma 2.3 *Let $\varphi(t, y) = \sup\{x \mid v(t, x) > y\} \vee 0$. Then*

$$\varphi(t, \mathcal{V}(t, y)) = \varphi_0(y)$$

whenever $0 < \mathcal{V}(t, y) < \bar{v}(t)$.

Proof: Using Proposition 2.2 we find that

$$\varphi(t, \mathcal{V}(t, y)) = \sup\{x \mid \mathcal{V}(t, v_0(x)) > \mathcal{V}(t, y)\} = \sup\{x \mid v_0(x) > y\} = \varphi_0(y),$$

provided $0 < \mathcal{V}(t, y) < \bar{v}(t)$. \square

3 Long-time behavior I

In this section we derive some basic results as time tends to infinity without scaling the solution. Below we shall denote the maximum particle volume at time t by

$$\bar{v}(t) = v(t, 0) = \sup_{\varphi} v(t, \varphi).$$

Note that from (7) it follows

$$\bar{v}(t)^{1/3} \theta(t) \geq 1,$$

therefore $\bar{v}(t)$ is *increasing*. The total volume of particles in the system will be written

$$V = \int_0^1 v(t, \varphi) d\varphi.$$

3.1 The case of a Dirac delta at the tip

First we consider the case in which a positive fraction of the particles have volume equal to the maximum possible in the system. In terms of the particle volume distribution f this means we consider measure-valued initial data f_0 that carry a Dirac delta at the tip of the support. That is,

$$f_0(v) = a\delta(v - \bar{v}_0) + \tilde{f}_0(v)$$

where $0 < a \leq 1$, $\delta(v - \bar{v}_0)$ denotes the Dirac delta distribution at \bar{v}_0 and

$$\lim_{v \rightarrow \bar{v}_0} \int_v^\infty \tilde{f}_0(y) dy = 0.$$

For the volume ranking function v this assumption means that the solution $v(t, \cdot)$ is constant on the interval $[0, a)$. In this case we can characterize the asymptotic behavior as follows.

Proposition 3.1 *Assume that for some $a \in (0, 1]$, the initial data satisfy $v_0(0) = v_0(\varphi)$ for $0 \leq \varphi < a$ and $v_0(0) > v_0(\varphi)$ for $\varphi > a$. Then*

$$\lim_{t \rightarrow \infty} \bar{\varphi}(t) = a,$$

and we have

$$\lim_{t \rightarrow \infty} v(t, \varphi) = \begin{cases} V/a, & 0 \leq \varphi < a, \\ 0, & \varphi > a. \end{cases}$$

Correspondingly,

$$\lim_{t \rightarrow \infty} \varphi(t, v) = \begin{cases} a, & 0 \leq v < V/a, \\ 0, & v > V/a. \end{cases}$$

Proof: From the assumption on the initial data it follows that for all t , $v(t, \varphi) = \bar{v}(t)$ for $0 \leq \varphi < a$ and $v(t, \varphi) < \bar{v}(t)$ for $\varphi > a$. From conservation of volume it follows

$$V = \int_0^\infty v(t, \varphi) d\varphi \geq a\bar{v}(t).$$

Hence $\bar{v}(t)$ is uniformly bounded, and since it is increasing the limit $v_\infty = \lim_{t \rightarrow \infty} \bar{v}(t)$ exists.

Assume now that

$$\lim_{t \rightarrow \infty} \bar{\varphi}(t) > a.$$

(Since $\bar{v}(t) \geq V > 0$, $\bar{\varphi}(t) \geq a$ for all t .) Then there exists $\varphi > a$ such that $v(t, \varphi) > 0$ for all $t > 0$. But then for almost every $t > 0$, using the inequality $a - b \geq (a^3 - b^3)/3a^2$ for $a > b > 0$ we have

$$\begin{aligned} \partial_t(\bar{v}(t) - v(t, \varphi)) &= \theta(t) \left(\bar{v}(t)^{1/3} - v(t, \varphi)^{1/3} \right) \\ &\geq \frac{\theta(t) \bar{v}(t)^{1/3}}{3\bar{v}(t)} (\bar{v}(t) - v(t, \varphi)) \\ &\geq \frac{a}{3V} (\bar{v}(t) - v(t, \varphi)), \end{aligned}$$

and hence

$$\bar{v}(t) - v(t, \varphi) \geq (\bar{v}(0) - v_0(\varphi)) e^{at/3V}.$$

Since by assumption $\bar{v}(0) - v_0(\varphi) > 0$ and since $\bar{v}(t)$ is bounded this contradicts the presumption that $v(t, \varphi) > 0$ for all t . It follows that for each $\varphi > a$, $v(t, \varphi) = 0$ for sufficiently large t , therefore $\lim_{t \rightarrow \infty} \bar{\varphi}(t) = a$.

That $v_\infty = \lim_{t \rightarrow \infty} \bar{v}(t) = V/a$ follows by volume conservation. The limiting behavior asserted for $\varphi(t, v)$ follows from the fact that $a \leq \varphi(t, v) \leq \bar{\varphi}(t)$ for $0 \leq v < \bar{v}(t)$ and $\varphi(t, v) = 0$ for $v > \bar{v}(t)$. \square

3.2 The case of no Dirac delta at the tip

Here and throughout the rest of this work we will assume that no positive fraction of the particles have the maximal particle volume \bar{v} . For the initial volume distribution f_0 this means

$$\lim_{v \rightarrow \bar{v}_0} \int_v^\infty f_0(y) dy = 0.$$

Proposition 3.2 *Assume that $v_0(0) > v_0(\varphi)$ for all $\varphi \in (0, 1]$. Then*

- (a) $\lim_{t \rightarrow \infty} \bar{v}(t) = \infty$, i.e., the maximal particle volume tends to infinity.
- (b) $\liminf_{t \rightarrow \infty} \theta(t) = 0$, meaning the critical radius is unbounded.
- (c) $\lim_{t \rightarrow \infty} \bar{\varphi}(t) = 0$, i.e., the fraction of initially existing particles that continue to exist at time t decreases to zero.

We do not know whether the result in (b) can be improved to say $\lim_{t \rightarrow \infty} \theta(t) = 0$.

Proof: We prove that $\bar{v}(t)$ is unbounded using volume conservation. Suppose that $\bar{v}(t) \leq C$ for all t . As in the proof of Proposition 3.1 we find that for any $\varphi > 0$, as long as $v(t, \varphi) > 0$ we have

$$\bar{v}(t) - v(t, \varphi) \geq (\bar{v}(0) - v_0(\varphi))e^{t/3C}.$$

Consequently, for each φ , $v(t, \varphi) = 0$ for sufficiently large t . This means that $\bar{\varphi}(t) \rightarrow 0$ as $t \rightarrow \infty$. Then we infer that $V \leq \bar{v}(t)\bar{\varphi}(t) \rightarrow 0$, so $V = 0$, but the hypotheses guarantee $V > 0$. Hence $\bar{v}(t)$ is unbounded; since it is increasing, it tends to infinity.

Suppose the claim in (b) fails. Then since $\theta(t)$ is bounded away from zero for t in any finite interval, there is some $\delta > 0$ such that

$$\theta(t) \geq \delta$$

for all t . From (a) and the fact that $\varphi \mapsto v(t, \varphi)$ is right continuous, there exists $\varepsilon > 0$ and (t_0, φ_0) with $\varphi_0 > 0$ such that $v(t_0, \varphi_0)^{1/3}\delta - 1 > \varepsilon$, which implies that

$$\frac{\partial v}{\partial t}(t, \varphi_0) = v^{1/3}\theta(t) - 1 \geq \varepsilon > 0$$

near $t = t_0$ and so for almost every $t \geq t_0$. This yields that $v(t, \varphi_0)$ is unbounded, and this contradicts the inequality

$$V = \int_0^1 v(t, \varphi) d\varphi \geq \varphi_0 v(t, \varphi_0), \tag{10}$$

which is valid for all t since $\varphi \mapsto v(t, \varphi)$ is decreasing. This proves (b).

To prove (c), note that since $v^{1/3} < 1 + v$ we have

$$\bar{\varphi}(t) = \theta(t) \int_0^1 v^{1/3}(t, \varphi) d\varphi \leq \theta(t)(1 + V).$$

By (b) there exists a sequence $t_n \rightarrow \infty$ such that $\theta(t_n) \rightarrow 0$. Since $\bar{\varphi}(t)$ is decreasing the assertion (c) follows. \square

4 Self-similar solutions

For understanding the long-time dynamics, we shall study the distribution function or particle rank φ as a function of volume v according to the dynamical formulation described in Lemma 2.3. (Recall that the usual distribution function $F(t, v) = 1 - \varphi(t, v)$.)

We look for solutions in self-similar form. We begin with some general considerations, temporarily allowing for the possibility that the volume distribution has infinite support. Any self-similar distribution function φ for which total particle volume is finite and conserved must have the form

$$\varphi(t, v) = a(t)\tilde{\psi}(a(t)v) \quad (11)$$

for some decreasing functions $\tilde{\psi}$ and a . The total volume satisfies

$$V = \int_0^\infty \varphi(t, v) dv = \int_0^\infty \tilde{\psi}(s) ds,$$

and we have

$$\theta(t) = \frac{\varphi(t, 0)}{\int_0^\infty \frac{1}{3}v^{-2/3}\varphi(t, v) dv} = \frac{a(t)^{1/3}\tilde{\psi}(0)}{\int_0^\infty \frac{1}{3}s^{-2/3}\tilde{\psi}(s) ds}. \quad (12)$$

Lemma 4.1 *Any self-similar distribution function φ in the form (11) with finite total volume has compact support.*

Proof: Given the form in (11), by (12) we may scale a and $\tilde{\psi}$ so that $\theta(t)a(t)^{-1/3} \equiv 1$. According to Lemma 2.3, being a solution requires that

$$a(t)\tilde{\psi}(a(t)\mathcal{V}(t, y)) = a(0)\tilde{\psi}(a(0)y)$$

whenever $0 < \mathcal{V}(t, y) < \bar{v}(t)$. For a nonconstant solution, a and $\tilde{\psi}$ should be Lipschitz, and differentiating we find that

$$0 = a'(t)(\tilde{\psi} + s\tilde{\psi}'(s)) + a^2(s^{1/3} - 1)\tilde{\psi}'(s).$$

Separating variables, we find that a'/a^2 must be a negative constant. We set $b = -a^2/a'$, then $a(t) = ba(0)/(b + a(0)t)$, and we must have

$$0 = \tilde{\psi}(s) + (s + b(1 - s^{1/3}))\tilde{\psi}'(s). \quad (13)$$

As long as $\tilde{\psi} > 0$ we compute that

$$-\ln \left(\frac{\tilde{\psi}(s)}{\tilde{\psi}(0)} \right) = \int_0^s \frac{dx}{x + b(1 - x^{1/3})}. \quad (14)$$

The function $s \mapsto s + b(1 - s^{1/3})$ is convex with the value $b > 0$ at $s = 0$, and at $s = (b/3)^{3/2}$ it achieves its minimum value $b(1 - \sqrt{4b/27})$. This is positive if $b < 27/4$ and nonpositive if $b \geq 27/4$.

In the former case, we have $\tilde{\psi}(s) > 0$ for all $s > 0$ and we compute that

$$\lim_{s \rightarrow \infty} -\ln \left(\frac{\tilde{\psi}(s)}{\tilde{\psi}(0)} \left(\frac{s+b}{b} \right) \right) = \int_0^\infty \left(\frac{1}{x + b(1 - x^{1/3})} - \frac{1}{x+b} \right) dx$$

and this is finite and positive. Hence $\tilde{\psi}(s) \sim C/s$ as $s \rightarrow \infty$ for some $C > 0$, and this means that the volume $\int_0^\infty \tilde{\psi}(s) ds$ diverges.

In the case $b \geq 27/4$, on the other hand, the integral in (14) diverges as $s \rightarrow s_0$, where s_0 is the smallest positive zero of $s + b(1 - s^{1/3})$. (This zero is between 1 and the value $(27/4)^{3/2}/3\sqrt{3} = 27/8$, and decreases to 1 as $b \rightarrow \infty$.) Then $\tilde{\psi}(s) \rightarrow 0$ as $s \rightarrow s_0$, and we must put $\tilde{\psi}(s) = 0$ for $s \geq s_0$. \square

Thus we see that all self-similar solutions with finite total volume have compact support. To describe these solutions in a way that connects with the analysis to come, we rescale with respect to the tip of the support, introducing the variables u and $\psi_*(u)$ that satisfy

$$1 - u = \frac{v}{\bar{v}(t)} = \frac{s}{s_0}, \quad \psi_*(u) = \frac{\bar{v}(t)\varphi(t, v)}{V} = \frac{s_0\tilde{\psi}(s)}{V}. \quad (15)$$

Note $v = s/a(t)$ and $\bar{v}(t) = s_0/a(t) = \bar{v}(0) + ts_0/b$. We set

$$\kappa_* = \frac{s_0 + b}{3s_0} = \frac{bs_0^{1/3}}{3s_0},$$

then $b/s_0 = 3\kappa_* - 1$ and $s_0 = (3\kappa_*/(3\kappa_* - 1))^3$, and κ_* increases from 1 to ∞ as b increases from $27/4$ to ∞ . Furthermore we have

$$\frac{s + b(1 - s^{1/3})}{s_0} = u(\kappa_* Q(u) - 1),$$

where

$$Q(u) = 3 \left(\frac{1 - (1 - u)^{1/3}}{u} \right). \quad (16)$$

The function Q satisfies $Q(0) = 1$, $Q(1) = 3$, and is given by the power series

$$Q(u) = 1 + \frac{1}{3}u + \frac{5}{27}u^2 + \dots,$$

which converges for $|u| < 1$ and whose coefficients are all positive.

The self-similar profiles $\psi_*(u)$ are increasing functions that satisfy

$$\frac{d}{du} \ln \psi_* = \frac{1}{u(\kappa_* Q(u) - 1)} \quad (17)$$

for $0 < u < 1$, with the normalization

$$\int_0^1 \psi_*(u) du = 1. \quad (18)$$

Note that for $\kappa_* > 1$,

$$\frac{1}{u(\kappa_* Q(u) - 1)} = \frac{1}{(\kappa_* - 1)u} - \left(\frac{\kappa_*}{\kappa_* - 1} \right) \frac{Q(u) - 1}{u(\kappa_* Q(u) - 1)}.$$

The last term is negative and remains bounded as $u \rightarrow 0$. Integrating (17), we find that self-similar solutions may be characterized according to the exponent $p = 1/(\kappa_* - 1)$, which governs the rate of vanishing as $u \rightarrow 0$. Henceforth, we write Ψ_p to denote the solution ψ_* of (17)–(18) with $\kappa_* = 1 + 1/p$.

Lemma 4.2 *For any $p \in (0, \infty]$, equation (9) for the distribution function or particle rank has a self-similar solution determined by (17) and (15) with $\kappa_* = 1 + p^{-1}$. For $p < \infty$ the profile has the form*

$$\Psi_p(u) = \alpha_p(u) u^p$$

where α_p is decreasing and analytic on $[0, 1)$. For $p = \infty$ ($\kappa_* = 1$) we have $\Psi_\infty(u) = o(u^q)$ as $u \rightarrow 0$ for all $q > 0$.

Equation (17) can be explicitly integrated using the substitution $1 - u = x^3$ and the expansion

$$u(\kappa_* Q(u) - 1) = (x - 1)(x - a)(x + a + 1)$$

where $2a = -1 + \sqrt{3(4\kappa_* - 1)}$. For $p < \infty$ ($\kappa_* > 1$) we find

$$\frac{\Psi_p(u)}{\Psi_p(1)} = \frac{(1 - x)^p}{\left(1 - \frac{x}{a}\right)^{p_1} \left(1 + \frac{x}{a+1}\right)^{p_2}} \quad (19)$$

where we have

$$p = \frac{3}{(a - 1)(a + 2)}, \quad p_1 = \frac{3a^2}{(a - 1)(2a + 1)}, \quad p_2 = \frac{3(a + 1)^2}{(a + 2)(2a + 1)}.$$

In ref. 8 analogous expressions are given for the corresponding particle distribution $\Phi(\tau, x) := \partial_u \Psi(\tau, u)$ where $1 - u = x^3$. The solution singled out by Lifshitz, Slyozov

and Wagner is that corresponding to $p = \infty$ or $\kappa_* = 1$ and satisfies

$$\frac{\Psi_\infty(u)}{\Psi_\infty(1)} = \frac{e^{-x/(1-x)}}{(1-x)^{5/3}(1+x/2)^{4/3}}. \quad (20)$$

When interpreting these formulae, note that

$$x = (1-u)^{1/3} = \left(\frac{v}{\bar{v}(t)}\right)^{1/3} = \frac{R}{\bar{R}(t)}$$

where R represents particle radius and $\bar{R}(t)$ is the maximal particle radius.

As $p \rightarrow 0$ in (19) we have $a \rightarrow \infty$ and we see that $\Psi_p(u) \rightarrow 1$ for $0 \leq u < 1$. That is, the particle distribution concentrates into a Dirac delta at the tip of the support. For the self-similar solution with profile Ψ_p , the maximal particle volume satisfies

$$\bar{v}(t) = \bar{v}(0) + \frac{pt}{2p+3}.$$

Thus, as Brown⁽⁵⁾ indicated, the rate of coarsening slows to zero as the particle distribution concentrates at the tip, which means $p \rightarrow 0$.

5 Long-time behavior II

5.1 Rescaled variables

In the following we always consider the case that no positive fraction of particles has the maximal volume \bar{v} . That is, we assume that the volume distribution carries no atom at the tip. To study the emergence of self-similar behavior for large times, it is appropriate to scale the volume by the maximal volume $\bar{v}(t)$. We introduce new variables via

$$\tau = \ln \frac{\bar{v}(t)}{\bar{v}(0)}, \quad u = 1 - \frac{v}{\bar{v}(t)}, \quad \psi = \frac{\bar{v}(t)\varphi}{V}. \quad (21)$$

For each $\tau \geq 0$, the function $u \mapsto \psi(\tau, u)$ is left continuous and increasing on $[0, 1]$ with $\psi(\tau, 0) = 0$. With $\psi_0(u) = \bar{v}(0)\varphi_0(v)/V$, Lemma 2.3 implies that the evolution of $\psi(\tau, u)$ is determined by the relation

$$\psi(\tau, \mathcal{U}(\tau, u)) = e^\tau \psi_0(u), \quad (22)$$

as long as $0 < \mathcal{U}(\tau, u) < 1$, where $\mathcal{U}(\tau, u) = 1 - \mathcal{V}(t, v)/\bar{v}(t)$ satisfies

$$\frac{\partial \mathcal{U}}{\partial \tau} = (\kappa(\tau)Q(\mathcal{U}) - 1)\mathcal{U}, \quad \mathcal{U}(0, u) = u. \quad (23)$$

Here Q is the function defined in (16), and $\kappa(\tau) > \frac{1}{3}$ is determined from

$$\frac{1}{\bar{v}(t)^{1/3}\theta(t)} = 1 - \frac{1}{3\kappa(\tau)} = \frac{1}{\psi(\tau, 1)} \int_0^1 \frac{1}{3}(1-u)^{-2/3}\psi(\tau, u) du. \quad (24)$$

The conservation of total volume is expressed by the identity

$$\int_0^1 \psi(\tau, u) du = 1, \quad (25)$$

valid for all $\tau \geq 0$.

For differentiable initial data, ψ satisfies

$$\partial_\tau \psi + (\kappa(\tau)Q(u) - 1)u\partial_u \psi = \psi \quad (26)$$

for $0 < u < 1$, $\tau > 0$. The characteristics of this PDE satisfy (23). Note that $\kappa(\tau)Q(1) - 1 > 0$ for all τ , and therefore every point $(\tau_1, u_1) \in (0, \infty) \times (0, 1)$ lies on a unique characteristic that can be continued back to time $\tau = 0$. That is, $u_1 = \mathcal{U}(\tau_1, \tilde{u}_1)$ for some $\tilde{u}_1 \in (0, 1)$.

Self-similar solutions of (9) correspond to stationary solutions of (26), for which we know $\kappa = \kappa_* \geq 1$ is constant and $\psi = \Psi_p$ where $p = 1/(\kappa_* - 1)$. In general we are interested in investigating under what conditions $\psi(\tau, u)$ converges as $\tau \rightarrow \infty$. We shall see that the long-time behavior of solutions is a sensitive function of the manner in which the initial data vanish as $u \rightarrow 0$.

5.2 Tip behavior and the mean field

Proposition 5.1 *Let ψ be determined by (22)–(25), and suppose $0 < p < \infty$.*

(a) *If $\inf_{u>0} \psi_0(u)/u^p > 0$, then $\limsup_{\tau \rightarrow \infty} \kappa(\tau) \geq 1 + \frac{1}{p}$.*

(b) *If $\sup_{u>0} \psi_0(u)/u^p < \infty$, then $\liminf_{\tau \rightarrow \infty} \kappa(\tau) \leq 1 + \frac{1}{p}$.*

Proof: By Lemma 4.2, given the hypotheses of part (a), there exists $a_0 > 0$ such that $\psi_0(u) \geq a_0 \Psi_p(u)$ for $0 < u < 1$. We aim to construct a “subsolution” of the form

$$\psi_-(\tau, u) = a(\tau)\Psi_p(u)$$

by taking $a(0) = a_0$ and ensuring that ψ/ψ_- is increasing along the characteristics given by (23). By (22) it is equivalent to show the map $\tau \mapsto \tau - \ln \psi_-(\tau, \mathcal{U}(\tau, u))$ is increasing. Using (17), we find that it suffices to ensure that $a(\cdot)$ is Lipschitz and for almost every $\tau > 0$ satisfies

$$\frac{a'(\tau)}{a(\tau)} \leq 1 - \frac{\kappa(\tau)Q(u) - 1}{\kappa_*Q(u) - 1} = \frac{\kappa_* - \kappa(\tau)}{\kappa_* - 1/Q(u)} \quad (27)$$

for $0 < u < 1$. Since $1 \leq Q(u) \leq 3$ this is guaranteed if we require

$$\frac{a'(\tau)}{a(\tau)} = \begin{cases} (\kappa_* - \kappa(\tau))/(\kappa_* - 1), & \text{if } \kappa_* - \kappa(\tau) > 0, \\ (\kappa_* - \kappa(\tau))/(\kappa_* - \frac{1}{3}), & \text{if } \kappa_* - \kappa(\tau) < 0, \\ 0, & \text{if } \kappa_* - \kappa(\tau) = 0. \end{cases} \quad (28)$$

With this choice, it follows $\psi(\tau, u)/a(\tau)\Psi_p(u) \geq 1$ for $0 \leq u \leq 1$ and for all $\tau \geq 0$.

Suppose now that $\limsup \kappa(\tau) < \kappa_*$. Then $\liminf a'(\tau)/a(\tau) > 0$ and it follows $a(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$. But this contradicts volume conservation. Hence $\limsup \kappa(\tau) \geq \kappa_*$.

Part (b) is proved in similar fashion. One constructs a supersolution of the form $\psi_+(\tau, u) = a(\tau)\Psi_p(u)$, ensuring that the inequality in (27) is reversed by interchanging the first two cases of (28). If $\liminf \kappa(\tau) > \kappa_*$ then $a(\tau) \rightarrow 0$ and again using volume conservation one obtains a contradiction. \square

5.3 Necessary conditions for convergence

Our aim in this section is to derive restrictions on initial data ψ_0 that must hold if convergence occurs in the rescaled variables.

5.3.1 Consequences of convergence

Lemma 5.2 *Assume that $\lim_{\tau \rightarrow \infty} \psi(\tau, u) = \psi_\infty(u)$ exists for each $u \in [0, 1]$. Then $\lim_{\tau \rightarrow \infty} 1/\kappa(\tau) = 1/\kappa_\infty$ exists, where $1 \leq \kappa_\infty \leq \infty$, and $\psi_\infty(u) = \Psi_p(u)$, where $p = 1/(\kappa_\infty - 1) \in [0, \infty]$. Moreover, as $\tau \rightarrow \infty$, $\psi(\tau, u)$ converges uniformly for u in any compact subset of $(0, 1]$.*

Proof: Since $\psi(\tau, u)$ is increasing in u , the dominated convergence theorem implies that $\lim_{\tau \rightarrow \infty} 1/\kappa(\tau)$ exists. If the limit is zero, then since ψ_∞ is increasing it follows from (24)–(25) that $\psi_\infty(u) = \psi_\infty(1) = 1$ for all $u \in (0, 1]$. This is the case $p = 0$ of the Lemma.

If the limit is nonzero, then $\frac{1}{3} \leq \kappa_\infty < \infty$. To show $\kappa_\infty \geq 1$, suppose otherwise. Then there is some $u_0 > 0$ for which $\kappa_\infty Q(u) - 1 < 0$ when $0 < u \leq u_0$. For sufficiently large τ this means that $\partial_\tau \mathcal{U} < 0$ if $\mathcal{U} \leq u_0$; thus there is a characteristic satisfying $0 < \mathcal{U}(\tau, u_1) < u_0$ for τ sufficiently large. Then (22) and the fact that $\psi(\tau, \cdot)$ is increasing yield that $\psi(\tau, u) \geq e^\tau \psi_0(u_1)$ for $u_0 \leq u \leq 1$, hence $\int_0^1 \psi(\tau, u) du \geq e^\tau (1 - u_0) \psi_0(u_1) \rightarrow \infty$ as $\tau \rightarrow \infty$. This contradicts volume conservation, so $\kappa_\infty \geq 1$.

Volume conservation also implies that $\psi_\infty(u_0) > 0$ for some $u_0 \in (0, 1)$. We aim to show that for any $u_1 \in (0, 1)$, $\psi_\infty(u_1) > 0$ and

$$\ln \psi_\infty(u_1) - \ln \psi_\infty(u_0) = \tau_{10} := \int_{u_0}^{u_1} \frac{du}{u(\kappa_\infty Q(u) - 1)}. \quad (29)$$

First, note that if we define \mathcal{U}_∞ to be the solution of

$$\frac{\partial \mathcal{U}_\infty}{\partial \tau} = (\kappa_\infty Q(\mathcal{U}_\infty) - 1)\mathcal{U}_\infty, \quad \mathcal{U}_\infty(0) = u_0,$$

then $\mathcal{U}_\infty(\tau)$ is defined for τ in a maximal interval of the form $(-\infty, T_0)$, with $\mathcal{U}_\infty(-\infty) = 0$, $\mathcal{U}(T_0) = 1$. Substituting $u = \mathcal{U}_\infty(\tau)$ into the integral in (29) reveals that $\mathcal{U}_\infty(\tau_{10}) = u_1$.

Consider the characteristic for (26) passing through u_0 at a large time τ_0 , as described by $u = \mathcal{U}_0(\tau)$ where

$$\frac{\partial \mathcal{U}_0}{\partial \tau} = (\kappa(\tau_0 + \tau)Q(\mathcal{U}_0) - 1)\mathcal{U}_0, \quad \mathcal{U}_0(0) = u_0.$$

Then since $\kappa(\tau) \rightarrow \kappa_\infty$ as $\tau \rightarrow \infty$ and Q is smooth on $(0, 1)$, standard arguments using Gronwall's inequality show that given any compact interval $[a, b] \subset (-\infty, T_0)$, we have

$$\sup_{a \leq \tau \leq b} |\mathcal{U}_0(\tau) - \mathcal{U}_\infty(\tau)| \rightarrow 0 \quad \text{as } \tau_0 \rightarrow \infty.$$

Fix such an interval that contains 0 and τ_{10} in its interior. Note that it follows that $\partial_\tau \mathcal{U}_0 > 0$ for $\tau \in [a, b]$ if τ_0 is sufficiently large.

From (22) it follows that for any fixed $\tau \in [a, b]$ we have

$$\psi(\tau_0 + \tau, \mathcal{U}_0(\tau)) = e^\tau \psi(\tau_0, u_0) \rightarrow e^\tau \psi_\infty(u_0) \quad \text{as } \tau_0 \rightarrow \infty.$$

If $\tau < \tau_{10}$, then for sufficiently large τ_0 we will have $\mathcal{U}_0(\tau) < \mathcal{U}_\infty(\tau_{10}) = u_1$, therefore $\psi(\tau_0 + \tau, \mathcal{U}_0(\tau)) \leq \psi(\tau_0 + \tau, u_1)$ and it follows $e^\tau \psi_\infty(u_0) \leq \psi_\infty(u_1)$. Similarly, if $\tau > \tau_{10}$ it follows $e^\tau \psi_\infty(u_0) \geq \psi_\infty(u_1)$. Therefore $e^{\tau_{10}} \psi_\infty(u_0) = \psi_\infty(u_1)$, and this proves the claim in (29).

It remains to show that ψ_∞ is continuous at $u = 1$, so that (29) holds also with $u_1 = 1$. Since Q is increasing, for some $\delta > 0$ we have $\partial_\tau \mathcal{U}_0 > \delta$ whenever $u_0 < \mathcal{U}_0 < 1$ and τ_0 is sufficiently large. Let $\varepsilon > 0$ be small. Then since $\psi(\tau, \cdot)$ is left continuous, for any sufficiently large τ_0 there exists u_0 close to 1 such that $\psi(\tau_0, u_0) > \psi_\infty(1) - \varepsilon$. With notation as above for this different value of u_0 we have $\mathcal{U}_0(-\varepsilon) < 1 - \delta\varepsilon$, and therefore

$$\psi(\tau_0 - \varepsilon, 1 - \delta\varepsilon) \geq \psi(\tau_0 - \varepsilon, \mathcal{U}_0(-\varepsilon)) = e^{-\varepsilon} \psi(\tau_0, u_0) \geq e^{-\varepsilon} (\psi_\infty(1) - \varepsilon)$$

The extreme members of these inequalities do not depend upon the choice of u_0 , so we can take $\tau_0 \rightarrow \infty$ and deduce that $\psi_\infty(1) \leq e^\varepsilon \psi_\infty(1 - \delta\varepsilon) + \varepsilon$. Since ε is arbitrary, we infer that ψ_∞ is continuous at 1, as required.

From (29) and the normalizations in (25) and (18) it now follows that $\psi_\infty(u) = \Psi_p(u)$ where $p = 1/(\kappa_\infty - 1)$, as claimed. Since Ψ_p is continuous and $\psi(\tau, \cdot)$ is increasing, a simple argument using pointwise convergence on a dense set in $[0, 1]$ shows that $\psi(\tau, u) \rightarrow \psi_\infty(u)$ uniformly in u . \square

As a consequence of Proposition 5.1 and Lemma 5.2, we find the following result, which implies that $\psi(\tau, u)$ fails to converge for a large set of initial data.

Corollary 5.3 *Assume that $\lim_{\tau \rightarrow \infty} \psi(\tau, u) = \psi_\infty(u)$ exists for each $u \in [0, 1]$. Then there exists $p \in [0, \infty]$ such that $\psi_\infty(u) = \Psi_p(u)$ and*

$$\inf_{u > 0} \psi_0(u)/u^q = 0 \quad \text{for all } q < p,$$

and

$$\sup_{u>0} \psi_0(u)/u^q = \infty \quad \text{for all } q > p.$$

These results imply the instability of the stationary solutions Ψ_p with respect to the sup norm topology, say, or any topology for which

- (a) the map from $\psi(\tau, \cdot)$ to $\kappa(\tau)$ is continuous for fixed τ , and
- (b) arbitrarily small perturbations of Ψ_p can yield initial data for which

$$\inf_{u>0} \frac{\psi_0(u)}{u^q} > 0 \quad \text{for some } q < p. \quad (30)$$

Corollary 5.4 *Given any $p \in (0, \infty]$, the stationary solution Ψ_p is unstable to perturbations that yield initial data that satisfy (30). In particular, the LSW solution (with $p = \infty$) is unstable to all perturbations for which $\inf_{u>0} \psi_0(u)/u^q > 0$ for some $q < \infty$.*

See section 6 for examples and further discussion.

5.3.2 Regularly varying functions

Corollary 5.3 leads one to expect that if $\psi(\tau, u) \rightarrow \Psi_p(u)$ for $0 \leq u \leq 1$ and if $p = 1/(\kappa_* - 1) < \infty$, then $\psi_0(u) \sim cu^p$ as $u \rightarrow 0$. This is almost but not quite exactly correct. In Theorem 5.10 below, we give a necessary criterion for convergence in this case that we conjecture is sharp. This criterion is related to the concept of regularly varying function as is treated in the books of Seneta⁽¹⁴⁾ and Bingham et al.,⁽²⁾ for example.

Definition 5.5 *A positive, measurable function g , defined on some interval of the form $(0, a]$, is called regularly varying at 0 (with exponent $p \in \mathbb{R}$) if*

$$\lim_{x \rightarrow 0^+} \frac{g(\lambda x)}{\lambda^p g(x)} = 1 \quad \text{for all } \lambda > 0. \quad (31)$$

If $p = 0$, we say g is slowly varying at 0.

Definition 5.6 We say that a real-valued, measurable function h , defined on some interval $[A, \infty)$, is locally linear at ∞ (with slant $p \in \mathbb{R}$) if

$$\lim_{y \rightarrow \infty} h(y + L) - h(y) = pL \quad \text{for all } L \in \mathbb{R}. \quad (32)$$

If $p = 0$ we say h is locally flat at ∞ .

It is a classic result (cf. Lemma 1.1 of ref. 14 or Theorem 1.2.1 of ref. 2) that if g is regularly varying at 0, then the limit in (31) is attained *uniformly* for all values of λ in any compact interval $[a, b] \subset (0, \infty)$. Similarly, if h is locally linear at ∞ , then the

limit in (32) is achieved uniformly for all values of L in any fixed finite interval. This fact implies we have the following characterization of the two definitions just given. We define the *oscillation* of a function h on an interval $[a, b]$ to be

$$\operatorname{osc}_{z \in [a, b]} h(z) = \sup_{z \in [a, b]} h(z) - \inf_{z \in [a, b]} h(z) = \sup_{z_1, z_2 \in [a, b]} h(z_1) - h(z_2).$$

Lemma 5.7 *Suppose $h(-\ln x) = -\ln g(x)$. Then the following are equivalent.*

- (i) g is regularly varying at 0 with exponent p .
- (ii) h is locally linear at ∞ with slant p .
- (iii) For all $L > 0$,

$$\operatorname{osc}_{z \in [y, y+L]} (h(z) - pz) \rightarrow 0 \quad \text{as } y \rightarrow \infty.$$

- (iv) For all $\lambda > 1$,

$$\frac{\sup_{z \in [x, \lambda x]} g(z)/z^p}{\inf_{z \in [x, \lambda x]} g(z)/z^p} \rightarrow 1 \quad \text{as } x \rightarrow 0.$$

The property of being regularly varying is invariant under a “nice” change of variables.

Lemma 5.8 *Suppose $x \mapsto X(x)$ is C^1 near $x = 0$ with $X(0) = 0$, $X' > 0$. Then, g is regularly varying at 0 with exponent p if and only if $g \circ X$ has the same property.*

Proof: It suffices to prove the “only if” part, since we can write $g = (g \circ X) \circ X^{-1}$ and X^{-1} has all the properties listed above for X . Assume that g is regularly varying at 0 with exponent p , and let $a < b$ with $[a, b] \subset (0, \infty)$. If $\lambda \in (a, b)$ is fixed we have

$$\tilde{\lambda}(x) := \lambda \frac{X(\lambda x)}{\lambda x} \frac{x}{X(x)} \rightarrow \lambda$$

as $x \rightarrow 0$, hence $\tilde{\lambda}(x) \in (a, b)$ for sufficiently small x . Then, from the fact that (31) holds uniformly for $\lambda \in [a, b]$ we infer that

$$\frac{g(X(\lambda x))}{\lambda^p g(X(x))} = \frac{g(\tilde{\lambda}(x)X(x))}{\tilde{\lambda}(x)^p g(X(x))} \left(\frac{\tilde{\lambda}(x)}{\lambda} \right)^p \rightarrow 1 \quad \text{as } x \rightarrow 0^+.$$

This proves that $g \circ X$ is regularly varying at 0 with exponent p . \square

5.3.3 Convergence requires regularly varying data

Recall from (22) that

$$\mathcal{U}(\tau, u) = 1 - \frac{\mathcal{V}(t, e^\tau \bar{v}(0)(1 - u))}{e^\tau \bar{v}(0)}$$

where $\tau = \ln(\bar{v}(t)/\bar{v}(0))$, and that $\mathcal{V}(t, \cdot)$ is analytic in a neighborhood of $\bar{v}(t)$ with positive derivative. From Lemma 5.8 and formula (22), we find the following.

Lemma 5.9 *For any $\tau > 0$ and $p \in [0, \infty)$, $\psi(\tau, \cdot)$ is regularly varying at 0 with exponent p if and only if ψ_0 has the same property.*

Here is our main result that gives a necessary condition for convergence in the rescaled variables to any one of the stationary solutions Ψ_p with $0 \leq p < \infty$. (This result does not address the case in which the limit is the LSW-solution, the case $p = \infty$ in Lemma 4.2.)

Theorem 5.10 *Assume that for some $p \in [0, \infty)$ we have*

$$\lim_{\tau \rightarrow \infty} \psi(\tau, u) = \Psi_p(u) \quad \text{for all } u \in [0, 1].$$

Then ψ_0 is regularly varying at 0 with exponent p .

Proof: By Lemma 5.9, to prove the theorem it suffices to show that for some $\tau_0 > 0$, $\psi(\tau_0, \cdot)$ is regularly varying at 0 with exponent p . We consider first the case $p > 0$. Since $\kappa(\tau) \rightarrow \kappa_*$ as $\tau \rightarrow \infty$, we may fix $\tau_0 > 0$ so that $|\kappa(\tau) - \kappa_*| \leq 1/2p$ for all $\tau \geq \tau_0$.

In the analysis to come we shall often make use of the variables

$$y = -\ln u, \quad \rho = -\ln \psi. \quad (33)$$

With $\rho_0(y) = -\ln \psi_0(u)$, we have

$$\rho(\tau, \mathcal{Y}(\tau, y)) = \rho_0(y) - \tau \quad (34)$$

whenever $\mathcal{Y}(\tau, y) > 0$, where $\mathcal{Y} = -\ln \mathcal{U}$ satisfies

$$\frac{\partial \mathcal{Y}}{\partial \tau} = -\kappa(\tau)Q(e^{-\mathcal{Y}}) + 1, \quad \mathcal{Y}(0, y) = y. \quad (35)$$

For differentiable initial data,

$$\partial_\tau \rho - (\kappa(\tau)Q(e^{-y}) - 1)\partial_y \rho = -1. \quad (36)$$

In these variables stationary solutions are described via $\rho_*(y) = -\ln \psi_*(u)$, and we have

$$(\kappa_*Q(e^{-y}) - 1)\rho'_*(y) = 1 \quad (37)$$

for all $y > 0$. For $0 < p = 1/(\kappa_* - 1) < \infty$ it follows

$$\rho_*(y) = py - A_*(y) \quad (38)$$

where $\lim_{y \rightarrow \infty} A_*(y)$ exists (and therefore A_* is locally flat at ∞).

Define

$$h(\tau, y) = \rho(\tau, y) - \rho_*(y). \quad (39)$$

The uniform convergence established in Lemma 5.2 implies that for any compact interval $[a, b] \subset [0, \infty)$, $\lim_{\tau \rightarrow \infty} h(\tau, y) = 0$ uniformly for $y \in [a, b]$.

To prove the theorem, it is enough to show that $h(\tau_0, \cdot)$ is locally flat at ∞ . This will be done by following pairs of characteristics backward to time τ_0 from large times $\tilde{\tau}$. The idea in the estimates that follow is that the total change in distance between such pairs can be bounded independently of $\tilde{\tau}$.

Let $\tilde{\mathcal{Y}}(\tau, \tilde{y})$ denote the characteristic satisfying

$$\frac{\partial \tilde{\mathcal{Y}}}{\partial \tau} = -(\kappa(\tau)Q(e^{-\tilde{\mathcal{Y}}}) - 1), \quad \tilde{\mathcal{Y}}(\tilde{\tau}, \tilde{y}) = \tilde{y}. \quad (40)$$

For $\tilde{\tau} \geq \tau \geq \tau_0$ we have

$$-\frac{\partial \tilde{\mathcal{Y}}}{\partial \tau} \geq \kappa(\tau) - 1 \geq \frac{1}{2p} \quad (41)$$

hence

$$\tilde{\mathcal{Y}}(\tau, \tilde{y}) \geq \tilde{y} + (\tilde{\tau} - \tau)/2p. \quad (42)$$

We also note that $\kappa(\tau) \leq K_0 := \kappa_* + 1/2p$ for $\tau \geq \tau_0$, so $\tilde{\mathcal{Y}}(\tau, \tilde{y}) \leq \tilde{y} + 3K_0(\tilde{\tau} - \tau)$.

Since $Q'(0) = 1/3$, there is some $\delta_0 > 0$ such that $Q'(e^{-y}) < 1$ whenever $y \geq \delta_0$. Suppose that $\delta_0 \leq \tilde{y} \leq \min(\tilde{y}_1, \tilde{y}_2)$. Let

$$\begin{aligned} \tilde{\mathcal{Y}}_1(\tau) &= \tilde{\mathcal{Y}}(\tau, \tilde{y}_1), & y_1 &= \tilde{\mathcal{Y}}_1(\tau_0), \\ \tilde{\mathcal{Y}}_2(\tau) &= \tilde{\mathcal{Y}}(\tau, \tilde{y}_2), & y_2 &= \tilde{\mathcal{Y}}_2(\tau_0), \end{aligned} \quad (43)$$

We compute that

$$E(\tilde{\tau}, \tilde{y}_1, \tilde{y}_2) := (y_1 - y_2) - (\tilde{y}_1 - \tilde{y}_2) = \int_{\tau_0}^{\tilde{\tau}} \kappa(\tau)(Q(e^{-\tilde{\mathcal{Y}}_1(\tau)}) - Q(e^{-\tilde{\mathcal{Y}}_2(\tau)})) d\tau. \quad (44)$$

Using (42) and the estimate $|Q(e^{-\tilde{\mathcal{Y}}_1}) - Q(e^{-\tilde{\mathcal{Y}}_2})| \leq e^{-\tilde{\mathcal{Y}}}$, we find that

$$|E(\tilde{\tau}, \tilde{y}_1, \tilde{y}_2)| \leq K_0 e^{-\tilde{y}} \int_{\tau_0}^{\tilde{\tau}} e^{-(\tilde{\tau}-\tau)/2p} d\tau \leq 2pK_0 e^{-\tilde{y}}. \quad (45)$$

From the relations (39), (34) and (37) we find that

$$\begin{aligned} H(\tilde{\tau}, \tilde{y}_1, \tilde{y}_2) &:= (h(\tau_0, y_1) - h(\tau_0, y_2)) - (h(\tilde{\tau}, \tilde{y}_1) - h(\tilde{\tau}, \tilde{y}_2)) \\ &= \int_{\tau_0}^{\tilde{\tau}} (\kappa_* - \kappa(\tau)) \frac{Q(e^{-\tilde{\mathcal{Y}}_2(\tau)}) - Q(e^{-\tilde{\mathcal{Y}}_1(\tau)})}{(\kappa_* Q(e^{-\tilde{\mathcal{Y}}_1(\tau)}) - 1)(\kappa_* Q(e^{-\tilde{\mathcal{Y}}_2(\tau)}) - 1)} d\tau. \end{aligned} \quad (46)$$

As above we have the estimate

$$|H| \leq M_0(\tilde{\tau}) := p^2 e^{-\tilde{y}} \int_{\tau_0}^{\tilde{\tau}} |\kappa_* - \kappa(\tau)| e^{-(\tilde{\tau}-\tau)/2p} d\tau. \quad (47)$$

Since $\kappa(\tau) - \kappa_* \rightarrow 0$, it follows $M_0(\tilde{\tau}) \rightarrow 0$ as $\tilde{\tau} \rightarrow \infty$.

Let $L > 0$, and fix an arbitrary $\tilde{y} > \delta_0$. Put $\tilde{y}_2 = \tilde{y}$ and choose $\tilde{y}_1 > \tilde{y}_2$ (depending on $\tilde{\tau}$) so that with $y = y(\tilde{\tau}) = y_2$ we have $y + L = y_1$. By (45) we have the bound $\tilde{y}_1 - \tilde{y}_2 \leq L + 2pK_0$, so we may deduce that $h(\tilde{\tau}, \tilde{y}_1) \rightarrow 0$ as $\tilde{\tau} \rightarrow \infty$.

Taking $\tilde{\tau} \rightarrow \infty$ in (46), we find that

$$\lim_{y \rightarrow \infty} h(\tau_0, y + L) - h(\tau_0, y) = 0. \quad (48)$$

Since $L > 0$ is arbitrary, it follows that $h(\tau_0, \cdot)$ is locally flat at ∞ . This finishes the proof of the theorem in the case $p > 0$.

It remains to consider the case $p = 0$, when $\Psi_p(u) = 1$ for $u \in (0, 1]$. In this case $\rho_* = 0$ and in (46) we have $H(\tilde{\tau}, \tilde{y}_1, \tilde{y}_2) = 0$. Moreover, $\lim_{\tau \rightarrow \infty} \kappa(\tau) = \infty$. We fix τ_0 sufficiently large so that $\kappa(\tau) > 2$ for $\tau \geq \tau_0$.

The strategy of the proof is the same as above. The key is to show that $E(\tilde{\tau}, \tilde{y}_1, \tilde{y}_2)$ is bounded independent of $\tilde{\tau}$ for $\tilde{\tau} > \tau_0$. Starting from (40), the first inequality in (41) remains valid, and moreover yields $\kappa(\tau) \leq -2\partial\tilde{Y}/\partial\tau$ for $\tau > \tau_0$. From (44) we get the bound

$$|E(\tilde{\tau}, \tilde{y}_1, \tilde{y}_2)| \leq \int_{\tau_0}^{\tilde{\tau}} \kappa(\tau) e^{-\tilde{Y}(\tau)} d\tau \leq 2 \int_{\tau_0}^{\tilde{\tau}} e^{-\tilde{Y}(\tau)} (-\partial\tilde{Y}/\partial\tau) d\tau \leq 2.$$

Arguing as before, we obtain (48) and the claimed result follows. \square

5.4 A sufficient condition for sufficiency

The estimates in the preceding proof can be used to establish some other interesting results. We conjecture that the condition that ψ_0 is regularly varying at 0 with exponent $p \geq 0$ is actually sufficient as well as necessary for convergence to Ψ_p as $\tau \rightarrow \infty$. A general proof is lacking, but in the subsection to follow, sufficiency is proved when $p > 0$ is sufficiently small and the initial data is close to Ψ_p in a sense related to the notion of regularly varying functions. The result given here reduces the general problem of sufficiency to proving the convergence of $\kappa(\tau)$.

Theorem 5.11 *Assume $\lim_{\tau \rightarrow \infty} \kappa(\tau) = \kappa_* \in (1, \infty]$. Then, $\lim_{\tau \rightarrow \infty} \psi(\tau, u)$ exists for all $u \in [0, 1]$ if and only if ψ_0 is regularly varying with exponent $p = 1/(\kappa_* - 1)$.*

Proof: The “only if” part has already been proved. Suppose ψ_0 is regularly varying with exponent p . We first consider the case $p > 0$. Then for $q < p$ we have $\psi_0(u)/u^q \rightarrow 0$ as $u \rightarrow 0$ and for $q > p$ we have $\psi_0(u)/u^q \rightarrow \infty$ as $u \rightarrow 0$. Since we assume $\kappa(\tau) \rightarrow \kappa_*$ as $\tau \rightarrow \infty$, Proposition 5.1 implies that $p = 1/(\kappa_* - 1)$. For sufficiently large τ_0 we have $\kappa(\tau) - 1 \geq 1/2p$ whenever $\tau \geq \tau_0$, and we can use the estimates from the proof of Theorem 5.10.

Our first step is to show that for any $L > 0$,

$$\sup_{z \in [\delta_0, \delta_0 + L]} |h(\tilde{\tau}, z) - b(\tilde{\tau})| \rightarrow 0 \quad \text{as } \tilde{\tau} \rightarrow \infty, \quad (49)$$

where we define $b(\tilde{\tau}) = h(\tilde{\tau}, \delta_0)$. To prove this we use (46), taking $\tilde{y}_2 = \delta_0$ and $\tilde{y}_1 \in [\delta_0, \delta_0 + L]$. Then $\tilde{\mathcal{Y}}_1 \geq \tilde{\mathcal{Y}}_2$ and it follows that $E(\tilde{\tau}, \tilde{y}_1, \tilde{y}_2) \leq 0$ (the characteristics are diverging), so we have

$$\delta_0 + (\tilde{\tau} - \tau_0)/2p \leq y_2 \leq y_1 \leq y_2 + L.$$

Now (46) yields that,

$$|h(\tilde{\tau}, \tilde{y}_1) - b(\tilde{\tau})| \leq \operatorname{osc}_{z \in [y_2, y_2 + L]} h(\tau_0, z) + M_0(\tilde{\tau}) \rightarrow 0$$

as $\tilde{\tau} \rightarrow \infty$ uniformly for $\tilde{y}_1 \in [\delta_0, \delta_0 + L]$. This proves (49).

Next, we show that for any $L > \delta_0$,

$$\sup_{z \in [0, L]} |h(\tilde{\tau}, z) - b(\tilde{\tau})| \rightarrow 0 \quad \text{as } \tilde{\tau} \rightarrow \infty. \quad (50)$$

To prove this, we use (42) taking $\tilde{y} = 0$, and use the identity (46) replacing τ_0 by $\tau_1 = \tilde{\tau} - 2p\delta_0$, which guarantees that $\tilde{\mathcal{Y}}(\tau_1, \tilde{y}) \geq \delta_0$. Fix $\tilde{y}_2 = \delta_0$. For any $\tilde{y}_1 \in [0, L]$ we have

$$\delta_0 \leq y_1 \leq \tilde{L} := L + 3K_0(\tilde{\tau} - \tau_1) = L + 6p\delta_0 K_0.$$

Then since $1 \leq Q \leq 3$ it follows from (46) that

$$|h(\tilde{\tau}, \tilde{y}_1) - b(\tilde{\tau})| \leq \operatorname{osc}_{z \in [\delta_0, \tilde{L}]} h(\tau_1, z) + 3p^2 \int_{\tau_1}^{\tilde{\tau}} |\kappa_* - \kappa(\tau)| d\tau$$

and this tends to zero as $\tilde{\tau} \rightarrow \infty$ uniformly for $\tilde{y}_1 \in [0, L]$, proving (50).

Now (50) implies that for any $L > 0$, $\rho(\tau, y) - b(\tau) \rightarrow \rho_*(y)$ as $\tau \rightarrow \infty$ uniformly for $y \in [0, L]$. Since $\rho(\tau, y) - b(\tau)$ increases to ∞ as $y \rightarrow \infty$ for each τ , we infer from the volume conservation identity (25) that as $\tau \rightarrow \infty$,

$$e^{b(\tau)} = \int_0^\infty e^{-\rho(\tau, y) + b(\tau)} e^{-y} dy \rightarrow \int_0^\infty e^{-\rho_*(y)} e^{-y} dy = 1.$$

Hence $\lim_{\tau \rightarrow \infty} b(\tau) = 0$. Then it quickly follows that $\lim_{\tau \rightarrow \infty} \psi(\tau, u)$ exists for each $u \in [0, 1]$. This finishes the proof when $p > 0$.

In the remaining case we assume $\lim_{\tau \rightarrow \infty} \kappa(\tau) = \infty$ and ψ_0 is regularly varying with exponent $p = 0$. Then $h(\tau_0, \cdot) = \rho(\tau_0, \cdot)$ is locally flat at ∞ for any $\tau_0 \geq 0$. We take τ_0 sufficiently large so that $\kappa(\tau) > 2$ for $\tau > \tau_0$, and take $\tilde{y}_2 = 0$ and $\tilde{y}_1 \in [0, L]$. Since $\tilde{\mathcal{Y}}_1(\tau) \geq \tilde{\mathcal{Y}}_2(\tau) \geq \tilde{\tau} - \tau$ for $\tau_0 \leq \tau \leq \tilde{\tau}$, it follows $E(\tilde{\tau}, \tilde{y}_1, \tilde{y}_2) \leq 0$, hence $\tilde{\tau} - \tau_0 \leq y_2 \leq y_1 \leq y_2 + L$. Since $H = 0$ in (46), we infer directly that

$$\sup_{z \in [0, L]} |h(\tilde{\tau}, z) - h(\tilde{\tau}, 0)| \leq \operatorname{osc}_{z \in [y_2, y_2 + L]} h(\tau_0, z) \rightarrow 0$$

as $\tilde{\tau} \rightarrow \infty$. With $b(\tau) = h(\tau, 0)$, this means that for all $L > 0$, $\rho(\tau, y) - b(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$ uniformly for $y \in [0, L]$. As above, from volume conservation we deduce that

$b(\tau) \rightarrow 0$ and hence $\lim_{\tau \rightarrow \infty} \psi(\tau, u) = 1$ for all $u \in (0, 1]$. \square

5.5 Stability and convergence for small p

In trying to establish convergence of $\psi(\tau, u)$ as $\tau \rightarrow \infty$ for regularly varying initial data, the main obstacle is achieving a priori control over $\kappa(\tau) - \kappa_*$. We can get such control when p is small, and obtain a stability result for the corresponding stationary solutions Ψ_p , with respect to perturbations that are small in a sense related to the notion of regularly varying functions.

For perturbations that are small in the appropriate sense, the condition that the initial data is regularly varying is necessary and sufficient for convergence as $\tau \rightarrow \infty$. In particular, the result below implies convergence to Ψ_p (for $p > 0$ small) for some initial data for which $\lim_{u \rightarrow 0} \psi_0(u)/u^p$ does not exist. If the initial data are not regularly varying, convergence cannot occur.

Recall that

$$h(\tau, y) = \rho(\tau, y) - \rho_*(y) = -\ln(\psi(\tau, e^{-y})/\Psi_p(e^{-y})).$$

We define the *flatness modulus* of h to be

$$\varpi(\tau, y) = \sup_{\tilde{y} \geq y} \operatorname{osc}_{z \in [\tilde{y}, \tilde{y}+1]} h(\tau, z). \quad (51)$$

Note that whenever $0 \leq a < b < c$,

$$\operatorname{osc}_{z \in [a, c]} h(\tau, z) \leq \operatorname{osc}_{z \in [a, b]} h(\tau, z) + \operatorname{osc}_{z \in [b, c]} h(\tau, z),$$

so for any positive integer n ,

$$\operatorname{osc}_{z \in [y, y+n]} h(\tau, z) \leq n\varpi(\tau, y). \quad (52)$$

Evidently, $\varpi(\tau, y)$ is decreasing in y , and we have $\varpi(\tau, y) \rightarrow 0$ as $y \rightarrow \infty$ if and only if $h(\tau, \cdot)$ is locally flat at ∞ .

Theorem 5.12 *For sufficiently small $p > 0$, there exist positive constants δ_* and K_0 with the following property. If the flatness modulus of h satisfies $\varpi(0, 0) \leq \delta_*$, then*

$$\varpi(\tau, 0) \leq K_0 \varpi(0, 0) \quad (53)$$

and

$$\sup_{0 < u \leq 1} |\psi(\tau, u) - \Psi_p(u)| \leq K_0 \varpi(\tau, 0) \quad (54)$$

for all $\tau \geq 0$.

If in addition ψ_0 is regularly varying at 0 with exponent p (equivalently, if $h(0, \cdot)$ is locally flat at ∞), then $\varpi(\tau, 0) \rightarrow 0$ as $\tau \rightarrow \infty$, and $\lim_{\tau \rightarrow \infty} \psi(\tau, u) = \Psi_p(u)$ for all

$u \in [0, 1]$.

Proof: We start by getting a bound on $|\kappa(\tau) - \kappa_*|$ where $\kappa_* = 1 + 1/p$. From (24), using $\psi_* = \Psi_p$ we have the representation

$$\begin{aligned} \frac{1}{\kappa(\tau)} - \frac{1}{\kappa_*} &= \int_0^1 \left(\frac{\psi_*(u)}{\psi_*(1)} - \frac{\psi(\tau, u)}{\psi(\tau, 1)} \right) (1-u)^{-2/3} du \\ &= \int_0^\infty \left(1 - e^{h(\tau, 0) - h(\tau, y)} \right) e^{\rho_*(0) - \rho_*(y)} (1 - e^{-y})^{-2/3} e^{-y} dy. \end{aligned} \quad (55)$$

We know that $\rho_*(y) - \rho_*(0) \geq 0$, and (52) implies

$$|h(\tau, 0) - h(\tau, y)| \leq (y+1)\varpi(\tau, 0). \quad (56)$$

So, since $|1 - e^x| \leq |x|e^{|x|}$ for all x , the integrand in (55) is bounded by

$$\varpi(\tau, 0)(y+1)e^{\varpi(\tau, 0)(y+1)}(1 - e^{-y})^{-2/3}e^{-y}.$$

Thus there is a constant \bar{K} independent of h and p such that if $\varpi(\tau, 0) \leq \frac{1}{2}$ then

$$\left| \frac{1}{\kappa(\tau)} - \frac{1}{\kappa_*} \right| \leq \bar{K}\varpi(\tau, 0). \quad (57)$$

It follows that if $\bar{K}\varpi(\tau, 0) \leq 1/4p\kappa_*^2 = p/4(p+1)^2$, then $\kappa(\tau) \leq 2\kappa_*$ and

$$|\kappa_* - \kappa(\tau)| \leq 2\kappa_*^2 \bar{K}\varpi(\tau, 0) \leq \frac{1}{2p}. \quad (58)$$

Next, we develop some a priori estimates for the flatness modulus, assuming it is small on some time interval.

Lemma 5.13 *For $p > 0$ there is a positive decreasing function $G_p: [0, \infty) \rightarrow \mathbb{R}$ such that*

$$\int_0^\infty G_p(\tau) d\tau \leq \bar{C}p(p+1)^2$$

for some constant $\bar{C} > 0$ independent of p , and the following holds. Assume that for some $\tau_ > 0$ we have $\bar{K}\varpi(\tilde{\tau}, 0) \leq p/4(p+1)^2$ for $0 \leq \tilde{\tau} \leq \tau_*$. Then*

$$\varpi(\tilde{\tau}, 0) \leq \varpi(0, \tilde{\tau}/2p) + \int_0^{\tilde{\tau}} G_p(\tilde{\tau} - \tau)\varpi(\tau, 0) d\tau$$

for $0 \leq \tilde{\tau} \leq \tau_$.*

Proof: We adopt the notation of the proof of Theorem 5.10. First, let $\tilde{y} \geq 0$ be arbitrary and suppose $\tilde{y}_1, \tilde{y}_2 \in [\tilde{y}, \tilde{y} + 1]$. We use the identity (46) and the fact that $1 \leq Q \leq 3$.

With any values of $\tau_0 \leq \tilde{\tau}$ between 0 and τ_* we have

$$|h(\tilde{\tau}, \tilde{y}_1) - h(\tilde{\tau}, \tilde{y}_2)| \leq |h(\tau_0, y_1) - h(\tau_0, y_2)| + 3p^2 \int_{\tau_0}^{\tilde{\tau}} |\kappa_* - \kappa(\tau)| d\tau.$$

Since the characteristics are diverging, y_1 and y_2 lie in $[\hat{y}, \hat{y} + 1]$ where $\hat{y} = \tilde{\mathcal{Y}}(\tau_0, \tilde{y}) \geq \tilde{y} + (\tilde{\tau} - \tau_0)/2p$, and so since $p^2\kappa_*^2 = (p+1)^2$, using (58) we find

$$\operatorname{osc}_{z \in [\hat{y}, \hat{y}+1]} h(\tilde{\tau}, z) \leq \operatorname{osc}_{z \in [\hat{y}, \hat{y}+1]} h(\tau_0, z) + 6(p+1)^2 \bar{K} \int_{\tau_0}^{\tilde{\tau}} \varpi(\tau, 0) d\tau.$$

Taking the sup over $\tilde{y} \geq y$ it follows that

$$\varpi(\tilde{\tau}, y) \leq \varpi(\tau_0, y + (\tilde{\tau} - \tau_0)/2p) + 6(p+1)^2 \bar{K} \int_{\tau_0}^{\tilde{\tau}} \varpi(\tau, 0) d\tau. \quad (59)$$

Next we use the same identity (46) but assume that $\tilde{y} \geq \delta_0$ as chosen in the proof of Theorem 5.10 (so $Q'(e^{-y}) < 1$ for $y > \delta_0$). Taking $\tau_0 = 0$, we obtain the estimate

$$|h(\tilde{\tau}, \tilde{y}_1) - h(\tilde{\tau}, \tilde{y}_2)| \leq |h(0, y_1) - h(0, y_2)| + p^2 e^{-\tilde{y}} \int_0^{\tilde{\tau}} |\kappa_* - \kappa(\tau)| e^{-(\tilde{\tau}-\tau)/2p} d\tau.$$

From this we deduce that for $y \geq \delta_0$,

$$\varpi(\tilde{\tau}, y) \leq \varpi(0, y + \tilde{\tau}/2p) + 2(p+1)^2 e^{-\delta_0} \bar{K} \int_0^{\tilde{\tau}} \varpi(\tau, 0) e^{-(\tilde{\tau}-\tau)/2p} d\tau. \quad (60)$$

Now, we invoke (59) with $y = 0$ and $\tau_0 = \max(0, \tilde{\tau} - 2p\delta_0)$, and replace $\tilde{\tau}$ in (60) by this τ_0 and use $y = \delta_0$. Then, with

$$G_p(\tau) = \begin{cases} 6(p+1)^2 \bar{K}, & 0 \leq \tau \leq 2p\delta_0, \\ 2(p+1)^2 \bar{K} e^{-\tau/2p}, & \tau \geq 2p\delta_0, \end{cases}$$

we find that the inequalities asserted in the Lemma are valid. \square

Now we proceed with the proof of Theorem 5.12. Assume p is sufficiently small so that $\bar{C}p(p+1)^2 \leq \frac{1}{2}$, and assume $\bar{K}\varpi(0, 0) < p/8(p+1)^2$. Let $M(\tilde{\tau}) = \sup_{0 \leq \tau \leq \tilde{\tau}} \varpi(\tau, 0)$, then using Lemma 5.13 and a simple continuation argument, we find that $M(\tilde{\tau}) \leq 2\varpi(0, 0)$ for all $\tilde{\tau} \geq 0$. Thus we may take $\delta_* = p/8\bar{K}(p+1)^2$ and $K_0 = 2$ to prove the first estimate (53) asserted in the Theorem.

The last part of the Theorem follows from Lemma 5.13 and the lemma below, together with the fact that if $h(0, \cdot)$ is locally flat at ∞ , then $\varpi(0, y) \rightarrow 0$ as $y \rightarrow \infty$. One deduces that $\varpi(\tau, 0) \rightarrow 0$ as $\tau \rightarrow \infty$. This implies $\kappa(\tau) \rightarrow \kappa_*$, and the convergence of $\psi(\tau, u)$ follows from Theorem 5.11.

Lemma 5.14 *Suppose f , a , and g are positive bounded measurable functions on $(0, \infty)$*

and $\int_0^\infty g(s) ds = \gamma < 1$. Assume

$$f(t) \leq a(t) + \int_0^t g(t-s)f(s) ds$$

for all $t \geq 0$, and $a(t) \rightarrow 0$ as $t \rightarrow \infty$. Then $f(t) \rightarrow 0$ as $t \rightarrow \infty$.

Proof: Let $F(t) = \sup_{0 \leq s \leq t} f(s)$, $A = \sup_{t \geq 0} a(t)$. Then $F(t) \leq A + \gamma F(t)$ so $F(t) \leq A/(1-\gamma)$ for all $t \geq 0$. Let $F_\infty = \limsup_{t \rightarrow \infty} f(t)$, and let $\varepsilon > 0$. Choose t_0 so $f(s) \leq F_\infty + \varepsilon$ whenever $s \geq t_0$. Then for $t \geq t_0$,

$$f(t) \leq a(t) + \frac{A}{1-\gamma} \int_0^{t_0} g(t-s) ds + \gamma(F_\infty + \varepsilon).$$

Taking the lim sup as $t \rightarrow \infty$, we find $F_\infty \leq \gamma(F_\infty + \varepsilon)$. Since ε is arbitrary, it follows $F_\infty = 0$. \square

It remains to prove the stability estimate (54). First, we get a bound on $|h(\tau, 0)|$. Since $\int_0^1 \psi(\tau, u) du = \int_0^1 \Psi_p(u) du = 1$ we find

$$e^{h(\tau, 0)} - 1 = \int_0^\infty \left(e^{h(\tau, 0) - h(\tau, y)} - 1 \right) \Psi_p(e^{-y}) e^{-y} dy.$$

Since $\Psi_p(u) \leq \alpha_p(0)u^p$ and $|e^x - 1| \leq |x|e^{|x|}$ it follows from (56) that provided $\varpi(\tau, 0) \leq \frac{1}{2}$ we have

$$\left| e^{h(\tau, 0)} - 1 \right| \leq \alpha_p(0) \varpi(\tau, 0) \int_0^\infty (y+1) e^{(y+1)\varpi(\tau, 0)} e^{-(p+1)y} dy \leq K_1 \varpi(\tau, 0)$$

where K_1 is a constant. Provided we can guarantee $\varpi(\tau, 0) \leq 1/2K_1$, we can ensure that $e^{h(\tau, 0)} \geq \frac{1}{2}$ and then

$$|h(\tau, 0)| = |\ln e^{h(\tau, 0)} - \ln e^0| \leq 2K_1 \varpi(\tau, 0). \quad (61)$$

From (56) we then infer $|h(\tau, y)| \leq (y + K_2) \varpi(\tau, 0)$ where $K_2 = 2K_1 + 1$. Then it follows that for $u = e^{-y} \in (0, 1]$,

$$|\psi(\tau, u) - \Psi_p(u)| \leq \left| e^{-h(\tau, y)} - 1 \right| \Psi_p(e^{-y}) \leq \alpha_p(0) |h(\tau, y)| e^{|h(\tau, y)| - py} \leq K_3 \varpi(\tau, 0) \quad (62)$$

where K_3 is a constant, provided we can guarantee that $\varpi(\tau, 0) \leq p/2$. By (53), the bounds on $\varpi(\tau, 0)$ that are needed to obtain (62) will hold if $\varpi(0, 0) \leq \delta_*$ for a sufficiently small value of δ_* . This finishes the proof of the Theorem. \square

6 Discussion

To recap and explain the main results of the last section, recall that $u = 1 - v/\bar{v}(t)$ and $\psi = (\bar{v}(t)/V)\varphi$ where $\varphi(t, x) = \int_x^\infty f(t, v) dv$, and suppose the initial data for ψ is written in the form

$$\psi_0(u) = \left(\frac{\bar{v}_0}{V}\right) \varphi_0(\bar{v}_0(1-u)) = \Psi_p(u) e^{-h_0(\ln(1/u))}. \quad (63)$$

Here ψ_0 must be increasing and is normalized so $\int_0^1 \psi_0(u) du = 1$. The results of section 5 show that:

- (i) If the scaled distribution function ψ converges pointwise as $\tau \rightarrow \infty$, then the limit is Ψ_p for some $p \in [0, \infty]$.
- (ii) If $\psi_0(u) \geq au^q > 0$ for some $q < \infty$ and $a > 0$, then ψ cannot converge to Ψ_∞ , the solution favored by the LSW theory.
- (iii) If ψ converges to Ψ_p for some $p \in [0, \infty)$, then ψ_0 is regularly varying at 0 with exponent p .
- (iv) If p is sufficiently small and positive, and if the flatness modulus of h_0 is sufficiently small, then ψ converges to Ψ_p *if and only if* ψ_0 is regularly varying at 0 with exponent p .

Recall that ψ_0 is regularly varying at 0 with exponent p if and only if the function h_0 in (63) is locally flat at ∞ . (Ψ_p can be replaced by u^p in (63) and this statement still holds.) Thus, if h_0 is bounded (implying $\psi_0(u) \geq au^p$) and is *not* locally flat at ∞ , then ψ cannot converge pointwise to *any* limit as $\tau \rightarrow \infty$. For a concrete example, any periodic or quasiperiodic function h_0 will suffice, e.g., $h_0 = a - b \sin y$, corresponding to $\psi_0(u) = Cu^p e^{b \sin \ln(1/u)}$.

If h_0 is C^1 , the mean value theorem implies that the flatness modulus of h_0 satisfies

$$\varpi_0(y) := \sup_{\tilde{y} \geq y} \operatorname{osc}_{z \in [\tilde{y}, \tilde{y}+1]} h_0(z) \leq \sup_{\tilde{y} \geq y} |h'_0(\tilde{y})|, \quad (64)$$

and so h_0 is locally flat at ∞ if $h'_0(y) \rightarrow 0$ as $y \rightarrow \infty$. As an example, suppose $h_0(y) = -\ln(1 + \varepsilon y) + c$, so $\psi_0(u) = C\Psi_p(u)(1 + \varepsilon \ln(1/u))$. Then h_0 is locally flat at ∞ and $\varpi_0(0) \leq \varepsilon$. So if p is small and ε is small, the corresponding solution ψ converges to Ψ_p in this case, despite the fact that $\psi_0(u)/u^p \rightarrow \infty$ as $u \rightarrow 0$.

Sharp representation formulae are known for regularly varying functions. For example, from Lemma 1.3 of ref. 14, it follows immediately that h_0 is locally flat at ∞ if and only if there exists $b > 0$ such that for $y > b$,

$$h_0(y) = f_1(y) + \int_b^y f_2(t) dt,$$

where f_1 and f_2 are bounded measurable functions such that $\lim_{y \rightarrow \infty} f_1(y)$ exists and $\lim_{y \rightarrow \infty} f_2(y) = 0$.

The results we have established in this paper show that the asymptotic behavior of solutions is exquisitely sensitive to the behavior of the initial data near the tip of the support. The conditions in (ii) and (iii) that ensure nonconvergence in the rescaled variables depend only on the initial size distribution of an arbitrarily small fraction of the largest particles. That is, they depend only on the initial volume ranking $v_0(\varphi)$ for values of the particle rank φ in an arbitrarily small interval $(0, \varepsilon_0)$.

It is clear that one can always perturb the sizes of an arbitrarily small fraction of the largest particles, by a maximum amount as small as you like, to guarantee that ψ_0 is not regularly varying at 0 and $\psi_0(u) \geq au^p$ for some p . This means that, with respect to the topology given by the sup norm distance between volume rankings (a natural topology for well-posedness of the initial value problem), there is a dense set of initial data that yield nonconvergence to self-similar form.

By similarly perturbing the sizes of the largest particles, one can guarantee that ψ_0 is regularly varying at 0 with any given exponent p . If it is true, as we conjecture, that having such data implies that ψ converges to the self-similar profile Ψ_p , then there is a dense set of initial data that yield convergence to Ψ_p , for any $p \in [0, \infty)$.

One can also make an arbitrarily small perturbation to make the new volume ranking constant on a small interval of the form $(0, \varepsilon)$. This means that the size distribution contains a Dirac delta at the tip, so by Proposition 3.1 the perturbed size distribution will converge to a stationary Dirac delta as $t \rightarrow \infty$.

We have not discussed in this paper any conditions on initial data for convergence to the smooth self-similar profile Ψ_∞ favored by the LSW theory. Lemma 5.2 implies that $\kappa(\tau) \rightarrow 1$ is necessary for such convergence. In this case, characteristics near the tip spread more slowly and different sorts of estimates seem to be needed to study the convergence question.

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